

OPTIMAL HARDY–LITTLEWOOD INEQUALITIES UNIFORMLY BOUNDED BY A UNIVERSAL CONSTANT

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ABSTRACT. The Hardy–Littlewood inequality for m -linear forms on ℓ_p spaces and $m < p \leq 2m$ asserts that

$$\left(\sum_{j_1, \dots, j_m=1}^{\infty} |T(e_{j_1}, \dots, e_{j_m})|^{\frac{p}{p-m}} \right)^{\frac{p-m}{p}} \leq 2^{\frac{m-1}{2}} \|T\|$$

for all continuous m -linear forms $T: \ell_p \times \dots \times \ell_p \rightarrow \mathbb{R}$ or \mathbb{C} . The case $m = 2$ recovers a classical inequality proved by Hardy and Littlewood in 1934. As a consequence of the results of the present paper we show that the same inequality is valid with $2^{\frac{m-1}{2}}$ replaced by $2^{\frac{(m-1)(p-m)}{p}}$. In particular, for $m < p \leq m+1$ the optimal constants of the above inequality are uniformly bounded by 2.

1. INTRODUCTION

The famous Littlewood’s 4/3 inequality [13], proved in 1930, asserts that

$$\left(\sum_{j,k=1}^{\infty} |T(e_j, e_k)|^{\frac{4}{3}} \right)^{\frac{3}{4}} \leq \sqrt{2} \|T\|$$

for all continuous bilinear forms $T: c_0 \times c_0 \rightarrow \mathbb{C}$, and the exponent 4/3 cannot be improved. Besides its own beauty, Littlewood’s insights motivated further important works of Bohnenblust and Hille (1931) and Hardy and Littlewood (1934). Bohnenblust–Hille inequality [7] assures the existence of a constant $B_m \geq 1$ such that

$$(1.1) \quad \left(\sum_{j_1, \dots, j_m=1}^{\infty} |T(e_{j_1}, \dots, e_{j_m})|^{\frac{2m}{m+1}} \right)^{\frac{m+1}{2m}} \leq B_m \|T\|,$$

for all continuous m -linear forms $T: c_0 \times \dots \times c_0 \rightarrow \mathbb{C}$. The case $m = 2$ recovers Littlewood’s 4/3 inequality. Three years later, using quite delicate estimates, Hardy and Littlewood [12] extended Littlewood’s 4/3 inequality to bilinear forms defined on $\ell_p \times \ell_q$. In 1981, Praciano-Pereira [19] extended the Hardy–Littlewood inequalities to m -linear forms on ℓ_p spaces for $p \geq 2m$ and quite recently Dimant and Sevilla-Peris [10] extended the estimates for the case $m < p \leq 2m$. These results were extensively investigated in various directions in the recent years ([1, 2, 4, 3, 5, 8, 10, 15]). As a matter of fact, all results hold for both real and complex scalars with eventually different constants; from now on we denote $\mathbb{K} = \mathbb{R}$ or \mathbb{C} . In general terms we have the following m -linear inequalities:

- If $p \geq 2m$, then there are constants $B_{m,p}^{\mathbb{K}} \geq 1$ such that

$$\left(\sum_{j_1, \dots, j_m=1}^n |T(e_{j_1}, \dots, e_{j_m})|^{\frac{2mp}{mp+p-2m}} \right)^{\frac{mp+p-2m}{2mp}} \leq B_{m,p}^{\mathbb{K}} \|T\|$$

for all m -linear forms $T: \ell_p^n \times \dots \times \ell_p^n \rightarrow \mathbb{K}$ and all positive integers n .

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- If $m < p \leq 2m$, then there are constants $B_{m,p}^{\mathbb{K}} \geq 1$ such that

$$\left(\sum_{j_1, \dots, j_m=1}^n |T(e_{j_1}, \dots, e_{j_m})|^{\frac{p}{p-m}} \right)^{\frac{p-m}{p}} \leq B_{m,p}^{\mathbb{K}} \|T\|$$

for all m -linear forms $T : \ell_p^n \times \dots \times \ell_p^n \rightarrow \mathbb{K}$ and all positive integers n .

The exponents of all above inequalities are optimal: if replaced by smaller exponents the constants will depend on n . However, looking at the above inequalities by an anisotropic viewpoint a much richer complexity arise (see, for instance, [1, 2, 4, 3, 8, 17]).

The investigation of the sharp constants in above inequalities is more than a puzzling mathematical challenge; for applications in physics we refer to [14]. The first estimates for $B_{m,p}^{\mathbb{K}}$ had exponential growth:

$$B_{m,p}^{\mathbb{K}} \leq (\sqrt{2})^{m-1},$$

for any $m \geq 1$. It was just quite recently that the estimates for $B_{m,p}^{\mathbb{K}}$ were refined, see for instance [4, 3, 6] and references therein. It was proved in [6] that

$$(1.2) \quad B_{m,\infty}^{\mathbb{R}} < \kappa_1 \cdot m^{\frac{2-\log 2-\gamma}{2}} \approx \kappa_1 \cdot m^{0.36482},$$

$$(1.3) \quad B_{m,\infty}^{\mathbb{C}} < \kappa_2 \cdot m^{\frac{1-\gamma}{2}} \approx \kappa_2 \cdot m^{0.21139},$$

for certain constants $\kappa_1, \kappa_2 > 0$, where γ is the Euler-Mascheroni constant. For $p < \infty$, among other results it was shown in [3] that for $p > 2m(m-1)^2$ we have

$$B_{m,p}^{\mathbb{K}} \leq B_{m,\infty}^{\mathbb{K}}.$$

The best known estimates of $B_{m,p}^{\mathbb{K}}$ for the case $m < p \leq 2m$ are $(\sqrt{2})^{m-1}$ (see [2, 10]). These estimates (case $m < p \leq 2m$) are somewhat intriguing. In fact, if $p = m$ it is easy to show that the only Hardy–Littlewood type inequality

$$\left(\sum_{j_1, \dots, j_m=1}^n |T(e_{j_1}, \dots, e_{j_m})|^s \right)^{\frac{1}{s}} \leq B_{m,m}^{\mathbb{K}} \|T\|$$

happens for $s = \infty$ (of course, here we consider the sup norm) and in this case it is obvious that the optimal constants are $B_{m,m}^{\mathbb{K}} = 1$. So, we have optimal constants equal to 1 for $p = m$ and the best known constants $(\sqrt{2})^{m-1}$ for p close to m . In this paper, among other results, we show that in fact the estimates $(\sqrt{2})^{m-1}$ are far from being optimal: we prove that

$$B_{m,p}^{\mathbb{K}} \leq 2^{\frac{(m-1)(p-m)}{p}}.$$

We present below the estimate obtained by Dimant and Sevilla-Peris ([10]) for further reference:

Theorem 1 (Dimant and Sevilla-Peris). *Let $m \geq 2$ be a positive integer and $p_j > 1$ for all j and*

$$\frac{1}{2} \leq \frac{1}{p_1} + \dots + \frac{1}{p_m} < 1.$$

Then

$$\left(\sum_{j_1, \dots, j_m=1}^n |T(e_{j_1}, \dots, e_{j_m})|^{\frac{1}{1-(\frac{1}{p_1} + \dots + \frac{1}{p_m})}} \right)^{1-(\frac{1}{p_1} + \dots + \frac{1}{p_m})} \leq (\sqrt{2})^{m-1} \|T\|,$$

for all m -linear forms $T : \ell_{p_1}^n \times \cdots \times \ell_{p_m}^n \rightarrow \mathbb{K}$ and all positive integers n . In particular, if $m < p \leq 2m$, then,

$$\left(\sum_{j_1, \dots, j_m=1}^n |T(e_{j_1}, \dots, e_{j_m})|^{\frac{p}{p-m}} \right)^{\frac{p-m}{p}} \leq (\sqrt{2})^{m-1} \|T\|$$

for all continuous m -linear forms $T : \ell_p \times \cdots \times \ell_p \rightarrow \mathbb{K}$.

The exponent $\frac{1}{1 - (\frac{1}{p_1} + \cdots + \frac{1}{p_m})}$ is optimal, but if one works in the anisotropic setting the result is not optimal (see, for instance, [5, 17]). The main results of the present paper are the forthcoming Theorems 2, 3 and 4 which also improve the original constants of the bilinear Hardy–Littlewood inequalities. For instance, for $m < p \leq m+1$ the optimal constants of the Hardy–Littlewood inequality are uniformly bounded by 2.

2. A MULTIPURPOSE LEMMA

Let $m \geq 2$ be a positive integer, F be a Banach space, $A \subset I_m := \{1, \dots, m\}$, $p_1, \dots, p_m, s, \alpha \geq 1$ and

$$B_{p_1, \dots, p_m}^{A, s, \alpha, F, n} := \inf \left\{ C(n) \geq 0 : \left(\sum_{j_i=1}^n \left(\sum_{\widehat{j_i=1}}^n |T(e_{j_1}, \dots, e_{j_m})|^s \right)^{\frac{1}{s}\alpha} \right)^{\frac{1}{\alpha}} \leq C(n), \text{ for all } i \in A \right\},$$

in which $\widehat{j_i}$ means that the sum runs over all indexes but j_i , and the infimum is taken over all norm-one m -linear operators $T : \ell_{p_1}^n \times \cdots \times \ell_{p_m}^n \rightarrow F$. The following lemma – fundamental in the proof of our main results – is based on ideas dating back to Hardy and Littlewood (see [12] and [19]), and we believe that it is of independent interest:

Lemma 1. *Let $1 \leq p_k < q_k \leq \infty$, $k = 1, \dots, m$ and $\lambda_0, s \geq 1$.*

(a) *If*

$$(2.1) \quad \sum_{j=1}^m \left(\frac{1}{p_j} - \frac{1}{q_j} \right) < \frac{1}{\lambda_0} \quad \text{and} \quad s \geq \left[\frac{1}{\lambda_0} - \sum_{j=1}^m \left(\frac{1}{p_j} - \frac{1}{q_j} \right) \right]^{-1} =: \eta_1,$$

then

$$B_{p_1, \dots, p_m}^{I_m, s, \eta_1, F, n} \leq B_{q_1, \dots, q_m}^{I_m, s, \lambda_0, F, n}.$$

(b) *If*

$$(2.2) \quad \sum_{j=1}^m \left(\frac{1}{p_j} - \frac{1}{q_j} \right) < \frac{1}{\lambda_0} \quad \text{and} \quad s \geq \left[\frac{1}{\lambda_0} - \sum_{j=1}^{m-1} \left(\frac{1}{p_j} - \frac{1}{q_j} \right) \right]^{-1} =: \eta_2$$

then

$$B_{p_1, \dots, p_m}^{\{m\}, s, \eta_2, F, n} \leq B_{q_1, \dots, q_m}^{I_m, s, \lambda_0, F, n}.$$

Proof. To prove (a), let s, λ_0 be such that (2.1) is fulfilled. Let us define

$$\lambda_j := \left[\frac{1}{\lambda_0} - \sum_{i=1}^j \left(\frac{1}{p_i} - \frac{1}{q_i} \right) \right]^{-1}, \quad j = 1, \dots, m.$$

Notice that $\lambda_m = \eta_1$,

$$\lambda_{j-1} < \lambda_j \quad \text{and} \quad \left[\frac{q_j p_j}{\lambda_{j-1}(q_j - p_j)} \right]^* = \frac{\lambda_j}{\lambda_{j-1}}, \quad \text{for all } j = 1, \dots, m.$$

Let us suppose that, for $k \in \{1, \dots, m\}$,

$$(2.3) \quad \left(\sum_{j_i=1}^n \left(\sum_{\widehat{j_i=1}}^n |T(e_{j_1}, \dots, e_{j_m})|^s \right)^{\frac{1}{s} \lambda_{k-1}} \right)^{\frac{1}{\lambda_{k-1}}} \leq B_{q_1, \dots, q_m}^{I_m, s, \lambda_0, F, n} \|T\|$$

is true for all continuous m -linear operators $T : \ell_{p_1}^n \times \dots \times \ell_{p_{k-1}}^n \times \ell_{q_k}^n \times \dots \times \ell_{q_m}^n \rightarrow F$ and for all $i = 1, \dots, m$. Let us prove that

$$(2.4) \quad \left(\sum_{j_i=1}^n \left(\sum_{\widehat{j_i=1}}^n |T(e_{j_1}, \dots, e_{j_m})|^s \right)^{\frac{1}{s} \lambda_k} \right)^{\frac{1}{\lambda_k}} \leq B_{q_1, \dots, q_m}^{I_m, s, \lambda_0, F, n} \|T\|$$

for all continuous m -linear operators $T : \ell_{p_1}^n \times \dots \times \ell_{p_k}^n \times \ell_{q_{k+1}}^n \times \dots \times \ell_{q_m}^n \rightarrow F$ and for all $i = 1, \dots, m$. The first induction step is our hypothesis. Consider

$$T : \ell_{p_1}^n \times \dots \times \ell_{p_k}^n \times \ell_{q_{k+1}}^n \times \dots \times \ell_{q_m}^n \rightarrow F,$$

a m -linear operator and, for each $x \in B_{\ell_{\frac{q_k p_k}{q_k - p_k}}^n}$ define

$$\begin{aligned} T^{(x)} : \ell_{p_1}^n \times \dots \times \ell_{p_{k-1}}^n \times \ell_{q_k}^n \times \dots \times \ell_{q_m}^n &\rightarrow F \\ (z^{(1)}, \dots, z^{(m)}) &\mapsto T(z^{(1)}, \dots, z^{(k-1)}, xz^{(k)}, z^{(k+1)}, \dots, z^{(m)}), \end{aligned}$$

with $xz^{(k)} = (x_j z_j^{(k)})_{j=1}^n \in \ell_{p_k}^n$. Observe that

$$\|T\| \geq \sup \left\{ \|T^{(x)}\| : x \in B_{\ell_{\frac{q_k p_k}{q_k - p_k}}^n} \right\}.$$

By applying the induction hypothesis to $T^{(x)}$, we obtain

$$\begin{aligned} &\left(\sum_{j_i=1}^n \left(\sum_{\widehat{j_i=1}}^n |T(e_{j_1}, \dots, e_{j_m})|^s |x_{j_k}|^s \right)^{\frac{1}{s} \lambda_{k-1}} \right)^{\frac{1}{\lambda_{k-1}}} \\ &= \left(\sum_{j_i=1}^n \left(\sum_{\widehat{j_i=1}}^n |T(e_{j_1}, \dots, e_{j_{k-1}}, x e_{j_k}, e_{j_{k+1}}, \dots, e_{j_m})|^s \right)^{\frac{1}{s} \lambda_{k-1}} \right)^{\frac{1}{\lambda_{k-1}}} \\ &= \left(\sum_{j_i=1}^n \left(\sum_{\widehat{j_i=1}}^n |T^{(x)}(e_{j_1}, \dots, e_{j_m})|^s \right)^{\frac{1}{s} \lambda_{k-1}} \right)^{\frac{1}{\lambda_{k-1}}} \\ &\leq B_{q_1, \dots, q_m}^{I_m, s, \lambda_0, F, n} \|T^{(x)}\| \\ (2.5) \quad &\leq B_{q_1, \dots, q_m}^{I_m, s, \lambda_0, F, n} \|T\| \end{aligned}$$

for all $i = 1, \dots, m$.

Since

$$\left[\frac{q_j p_j}{\lambda_{j-1}(q_j - p_j)} \right]^* = \frac{\lambda_j}{\lambda_{j-1}},$$

for all $j = 1, \dots, m$, we have

$$\begin{aligned}
& \left(\sum_{j_k=1}^n \left(\sum_{\hat{j}_k=1}^n |T(e_{j_1}, \dots, e_{j_m})|^s \right)^{\frac{1}{s} \lambda_k} \right)^{\frac{1}{\lambda_k}} \\
&= \left(\sum_{j_k=1}^n \left(\sum_{\hat{j}_k=1}^n |T(e_{j_1}, \dots, e_{j_m})|^s \right)^{\frac{1}{s} \lambda_{k-1} \left[\frac{q_k p_k}{\lambda_{k-1}(q_k - p_k)} \right]^*} \right)^{\frac{1}{\lambda_{k-1}} \cdot \frac{1}{\left[\frac{q_k p_k}{\lambda_{k-1}(q_k - p_k)} \right]^*}} \\
&= \left\| \left(\left(\sum_{\hat{j}_k=1}^n |T(e_{j_1}, \dots, e_{j_m})|^s \right)^{\frac{1}{s} \lambda_{k-1}} \right)^n \right\|_{j_k=1}^{\frac{1}{\lambda_{k-1}}} \left[\frac{q_k p_k}{\lambda_{k-1}(q_k - p_k)} \right]^* \\
&= \left(\sup_{y \in B_{\ell^n}^{\frac{q_k p_k}{\lambda_{k-1}(q_k - p_k)}}} \sum_{j_k=1}^n |y_{j_k}| \left(\sum_{\hat{j}_k=1}^n |T(e_{j_1}, \dots, e_{j_m})|^s \right)^{\frac{1}{s} \lambda_{k-1}} \right)^{\frac{1}{\lambda_{k-1}}} \\
&= \left(\sup_{x \in B_{\ell^n}^{\frac{q_k p_k}{q_k - p_k}}} \sum_{j_k=1}^n |x_{j_k}|^{\lambda_{k-1}} \left(\sum_{\hat{j}_k=1}^n |T(e_{j_1}, \dots, e_{j_m})|^s \right)^{\frac{1}{s} \lambda_{k-1}} \right)^{\frac{1}{\lambda_{k-1}}} \\
&= \sup_{x \in B_{\ell^n}^{\frac{q_k p_k}{q_k - p_k}}} \left(\sum_{j_k=1}^n \left(\sum_{\hat{j}_k=1}^n |T(e_{j_1}, \dots, e_{j_m})|^s |x_{j_k}|^s \right)^{\frac{1}{s} \lambda_{k-1}} \right)^{\frac{1}{\lambda_{k-1}}} \\
&\leq B_{q_1, \dots, q_m}^{I_{m,s,\lambda_0,F,n}} \|T\|.
\end{aligned}$$

This proves (2.4) for $i = k$. To prove (2.4) for $i \neq k$ let us consider initially $k \neq m$. Define

$$S_i = \left(\sum_{\hat{j}_i=1}^n |T(e_{j_1}, \dots, e_{j_m})|^s \right)^{\frac{1}{s}}, \quad i = 1, \dots, m.$$

Note that

$$\begin{aligned}
& \sum_{j_i=1}^n \left(\sum_{\widehat{j}_i=1}^n |T(e_{j_1}, \dots, e_{j_m})|^s \right)^{\frac{1}{s}\lambda_k} \\
&= \sum_{j_i=1}^n S_i^{\lambda_k} = \sum_{j_i=1}^n S_i^{\lambda_k-s} S_i^s \\
&= \sum_{j_i=1}^n \sum_{\widehat{j}_i=1}^n \frac{|T(e_{j_1}, \dots, e_{j_m})|^s}{S_i^{s-\lambda_k}} \\
&= \sum_{j_k=1}^n \sum_{\widehat{j}_k=1}^n \frac{|T(e_{j_1}, \dots, e_{j_m})|^s}{S_i^{s-\lambda_k}} \\
&= \sum_{j_k=1}^n \sum_{\widehat{j}_k=1}^n \frac{|T(e_{j_1}, \dots, e_{j_m})|^{\frac{s(s-\lambda_k)}{s-\lambda_{k-1}}}}{S_i^{s-\lambda_k}} |T(e_{j_1}, \dots, e_{j_m})|^{\frac{s(\lambda_k-\lambda_{k-1})}{s-\lambda_{k-1}}}.
\end{aligned}$$

From Hölder's inequality (first with exponents $r = \frac{s-\lambda_{k-1}}{s-\lambda_k}$ and $r^* = \frac{s-\lambda_{k-1}}{\lambda_k-\lambda_{k-1}}$ and then with exponents $r = \frac{\lambda_k(s-\lambda_{k-1})}{\lambda_{k-1}(s-\lambda_k)}$ and $r^* = \frac{\lambda_k(s-\lambda_{k-1})}{s(\lambda_k-\lambda_{k-1})}$), we have

$$\begin{aligned}
& \sum_{j_i=1}^n \left(\sum_{\widehat{j}_i=1}^n |T(e_{j_1}, \dots, e_{j_m})|^s \right)^{\frac{1}{s}\lambda_k} \\
&= \sum_{j_k=1}^n \sum_{\widehat{j}_k=1}^n \frac{|T(e_{j_1}, \dots, e_{j_m})|^{\frac{s(s-\lambda_k)}{s-\lambda_{k-1}}}}{S_i^{s-\lambda_k}} |T(e_{j_1}, \dots, e_{j_m})|^{\frac{s(\lambda_k-\lambda_{k-1})}{s-\lambda_{k-1}}} \\
&\leq \sum_{j_k=1}^n \left(\sum_{\widehat{j}_k=1}^n \frac{|T(e_{j_1}, \dots, e_{j_m})|^s}{S_i^{s-\lambda_{k-1}}} \right)^{\frac{s-\lambda_k}{s-\lambda_{k-1}}} \left(\sum_{\widehat{j}_k=1}^n |T(e_{j_1}, \dots, e_{j_m})|^s \right)^{\frac{\lambda_k-\lambda_{k-1}}{s-\lambda_{k-1}}} \\
&\leq \left(\sum_{j_k=1}^n \left(\sum_{\widehat{j}_k=1}^n \frac{|T(e_{j_1}, \dots, e_{j_m})|^s}{S_i^{s-\lambda_{k-1}}} \right)^{\frac{\lambda_k}{\lambda_{k-1}}} \right)^{\frac{\lambda_{k-1}}{\lambda_k} \cdot \frac{s-\lambda_k}{s-\lambda_{k-1}}} \\
&\quad \times \left(\sum_{j_k=1}^n \left(\sum_{\widehat{j}_k=1}^n |T(e_{j_1}, \dots, e_{j_m})|^s \right)^{\frac{1}{s}\lambda_k} \right)^{\frac{1}{\lambda_k} \cdot \frac{(\lambda_k-\lambda_{k-1})s}{s-\lambda_{k-1}}}
\end{aligned} \tag{2.6}$$

Let us estimate separately the two factors of this product. It follows from the case $i = k$ that

$$\left(\sum_{j_k=1}^n \left(\sum_{\widehat{j}_k=1}^n |T(e_{j_1}, \dots, e_{j_m})|^s \right)^{\frac{1}{s}\lambda_k} \right)^{\frac{1}{\lambda_k} \cdot \frac{(\lambda_k-\lambda_{k-1})s}{s-\lambda_{k-1}}} \leq \left(B_{q_1, \dots, q_m}^{I_m, s, \lambda_0, F, n} \|T\| \right)^{\frac{(\lambda_k-\lambda_{k-1})s}{s-\lambda_{k-1}}}. \tag{2.7}$$

For the first factor, from Hölder's inequality with exponents $r = \frac{s}{s-\lambda_{k-1}}$ and $r^* = \frac{s}{\lambda_{k-1}}$ and the induction hypothesis, we get

$$\begin{aligned}
& \left(\sum_{j_k=1}^n \left(\sum_{\widehat{j_k}=1}^n \frac{|T(e_{j_1}, \dots, e_{j_m})|^s}{S_i^{s-\lambda_{k-1}}} \right)^{\frac{\lambda_k}{\lambda_{k-1}}} \right)^{\frac{\lambda_{k-1}}{\lambda_k}} \\
&= \left\| \left(\sum_{\widehat{j_k}} \frac{|T(e_{j_1}, \dots, e_{j_m})|^s}{S_i^{s-\lambda_{k-1}}} \right)_{j_k=1}^n \right\|_{\left[\frac{q_k p_k}{\lambda_{k-1}(q_k - p_k)} \right]^*}^n \\
&= \sup_{y \in B_{\ell^n}^{\frac{q_k p_k}{\lambda_{k-1}(q_k - p_k)}}} \sum_{j_k=1}^n |y_{j_k}| \sum_{\widehat{j_k}=1}^n \frac{|T(e_{j_1}, \dots, e_{j_m})|^s}{S_i^{s-\lambda_{k-1}}} \\
&= \sup_{x \in B_{\ell^n}^{\frac{q_k p_k}{q_k - p_k}}} \sum_{j_k=1}^n |x_{j_k}|^{\lambda_{k-1}} \sum_{\widehat{j_k}=1}^n \frac{|T(e_{j_1}, \dots, e_{j_m})|^s}{S_i^{s-\lambda_{k-1}}} \\
&= \sup_{x \in B_{\ell^n}^{\frac{q_k p_k}{q_k - p_k}}} \sum_{j_k=1}^n \sum_{\widehat{j_k}=1}^n \frac{|T(e_{j_1}, \dots, e_{j_m})|^s}{S_i^{s-\lambda_{k-1}}} |x_{j_k}|^{\lambda_{k-1}} \\
&= \sup_{x \in B_{\ell^n}^{\frac{q_k p_k}{q_k - p_k}}} \sum_{j_i=1}^n \sum_{\widehat{j_i}=1}^n \frac{|T(e_{j_1}, \dots, e_{j_m})|^s}{S_i^{s-\lambda_{k-1}}} |x_{j_k}|^{\lambda_{k-1}} \\
&= \sup_{x \in B_{\ell^n}^{\frac{q_k p_k}{q_k - p_k}}} \sum_{j_i=1}^n \sum_{\widehat{j_i}=1}^n \frac{|T(e_{j_1}, \dots, e_{j_m})|^{s-\lambda_{k-1}}}{S_i^{s-\lambda_{k-1}}} |T(e_{j_1}, \dots, e_{j_m})|^{\lambda_{k-1}} |x_{j_k}|^{\lambda_{k-1}} \\
&\leq \sup_{x \in B_{\ell^n}^{\frac{q_k p_k}{q_k - p_k}}} \sum_{j_i=1}^n \left(\sum_{\widehat{j_i}=1}^n \frac{|T(e_{j_1}, \dots, e_{j_m})|^s}{S_i^s} \right)^{\frac{s-\lambda_{k-1}}{s}} \left(\sum_{\widehat{j_i}=1}^n |T(e_{j_1}, \dots, e_{j_m})|^s |x_{j_k}|^s \right)^{\frac{1}{s} \lambda_{k-1}} \\
&= \sup_{x \in B_{\ell^n}^{\frac{q_k p_k}{q_k - p_k}}} \sum_{j_i=1}^n \left(\sum_{\widehat{j_i}=1}^n |T(e_{j_1}, \dots, e_{j_m})|^s |x_{j_k}|^s \right)^{\frac{1}{s} \lambda_{k-1}} \\
&= \sup_{x \in B_{\ell^n}^{\frac{q_k p_k}{q_k - p_k}}} \sum_{j_i=1}^n \left(\sum_{\widehat{j_i}=1}^n |T^{(x)}(e_{j_1}, \dots, e_{j_m})|^s \right)^{\frac{1}{s} \lambda_{k-1}} \\
&\leq \left(B_{q_1, \dots, q_m}^{I_m, s, \lambda_0, F, n} \|T\| \right)^{\lambda_{k-1}}.
\end{aligned}$$

Therefore,

$$(2.8) \quad \left(\sum_{j_k=1}^n \left(\sum_{\widehat{j_k}=1}^n \frac{|T(e_{j_1}, \dots, e_{j_m})|^s}{S_i^{s-\lambda_{k-1}}} \right)^{\frac{\lambda_k}{\lambda_{k-1}}} \right)^{\frac{\lambda_{k-1}}{\lambda_k}} \leq \left(B_{q_1, \dots, q_m}^{I_m, s, \lambda_0, F, n} \|T\| \right)^{\lambda_{k-1} \cdot \frac{s-\lambda_k}{s-\lambda_{k-1}}}.$$

Replacing (2.7) and (2.8) in (2.6) we finally conclude that

$$\begin{aligned}
& \sum_{j_i=1}^n \left(\sum_{\widehat{j_i=1}}^n |T(e_{j_1}, \dots, e_{j_m})|^s \right)^{\frac{1}{s}\lambda_k} \\
& \leq (B_{q_1, \dots, q_m}^{I_m, s, \lambda_0, F, n} \|T\|)^{\lambda_{k-1} \cdot \frac{s-\lambda_k}{s-\lambda_{k-1}}} \cdot (B_{q_1, \dots, q_m}^{I_m, s, \lambda_0, F, n} \|T\|)^{\frac{(\lambda_k - \lambda_{k-1})s}{s-\lambda_{k-1}}} \\
& = (B_{q_1, \dots, q_m}^{I_m, s, \lambda_0, F, n})^{\lambda_{k-1} \cdot \frac{s-\lambda_k}{s-\lambda_{k-1}}} \cdot (B_{q_1, \dots, q_m}^{I_m, s, \lambda_0, F, n})^{\frac{(\lambda_k - \lambda_{k-1})s}{s-\lambda_{k-1}}} \cdot \|T\|^{\lambda_{k-1} \cdot \frac{s-\lambda_k}{s-\lambda_{k-1}}} \cdot \|T\|^{\frac{(\lambda_k - \lambda_{k-1})s}{s-\lambda_{k-1}}} \\
& = (B_{q_1, \dots, q_m}^{I_m, s, \lambda_0, F, n})^{\lambda_k} \|T\|^{\lambda_k},
\end{aligned}$$

that is,

$$\left(\sum_{j_i=1}^n \left(\sum_{\widehat{j_i=1}}^n |T(e_{j_1}, \dots, e_{j_m})|^s \right)^{\frac{1}{s}\lambda_k} \right)^{\frac{1}{\lambda_k}} \leq B_{q_1, \dots, q_m}^{I_m, s, \lambda_0, F, n} \|T\|.$$

It remains to consider $k = m$, where $\lambda_m = \eta_1$. In this case we have

$$\begin{aligned}
\left(\sum_{j_i=1}^n \left(\sum_{\widehat{j_i=1}}^n |T(e_{j_1}, \dots, e_{j_m})|^s \right)^{\frac{1}{s}\eta_1} \right)^{\frac{1}{\eta_1}} &= \left(\sum_{j_m=1}^n \left(\sum_{\widehat{j_m=1}}^n |T(e_{j_1}, \dots, e_{j_m})|^s \right)^{\frac{1}{s}\eta_1} \right)^{\frac{1}{\eta_1}} \\
&\leq B_{q_1, \dots, q_m}^{I_m, s, \lambda_0, F, n} \|T\|,
\end{aligned}$$

where the inequality is due to the case $i = k$.

The proof of (b) is similar, except for the last step (case $k = m$), but the argument is somewhat predictable and we omit the proof. \square

Remark 1. The case $q_k = \infty$ for all $k = 1, \dots, m$ in (a) is known; see, for instance, [10].

3. MAIN RESULTS

We begin with a technical lemma based on the Contraction Principle (see [9, Theorem 12.2]). From now on $r_i(t)$ are the Rademacher functions.

Lemma 2. *Regardless of the choice of the positive integers m, N and the scalars a_{i_1, \dots, i_m} , $i_1, \dots, i_m = 1, \dots, N$,*

$$\max_{\substack{i_k=1, \dots, N \\ k=1, \dots, m}} |a_{i_1, \dots, i_m}| \leq \int_{[0,1]^m} \left| \sum_{i_1, \dots, i_m=1}^N r_{i_1}(t_1) \cdots r_{i_m}(t_m) a_{i_1, \dots, i_m} \right| dt_1 \cdots dt_m.$$

Proof. Essentially, one just need to apply the Contraction Principle successively. We proceed by induction over m . The case $m = 1$ is precisely the standard version of Contraction Principle.

For all positive integers i_1, \dots, i_m ,

$$\begin{aligned}
& \int_{[0,1]^m} \left| \sum_{i_1, \dots, i_m=1}^N r_{i_1}(t_1) \cdots r_{i_m}(t_m) a_{i_1, \dots, i_m} \right| dt_1 \cdots dt_m \\
&= \int_{[0,1]^{m-1}} \left[\int_0^1 \left| \sum_{i_1=1}^N r_{i_1}(t_1) \left(\sum_{i_2, \dots, i_m=1}^N r_{i_2}(t_2) \cdots r_{i_m}(t_m) a_{i_1, \dots, i_m} \right) \right| dt_1 \right] dt_2 \cdots dt_m \\
&\geq \int_{[0,1]^{m-1}} \left| \sum_{i_2, \dots, i_m=1}^N r_{i_2}(t_2) \cdots r_{i_m}(t_m) a_{i_1, \dots, i_m} \right| dt_2 \cdots dt_m \\
&\geq |a_{i_1, \dots, i_m}|,
\end{aligned}$$

where we used the Contraction Principle and the induction hypothesis on the first and second inequality, respectively. This concludes the proof. \square

Now we are able to prove our first main result, providing better constants for Theorem 1:

Theorem 2. *Let $m \geq 2$ be a positive integer, $p_j > 1$ for all j and*

$$\frac{1}{2} \leq \frac{1}{p_1} + \cdots + \frac{1}{p_m} < 1.$$

Then

$$\left(\sum_{j_1, \dots, j_m=1}^n |T(e_{j_1}, \dots, e_{j_m})|^{\frac{1}{1 - (\frac{1}{p_1} + \cdots + \frac{1}{p_m})}} \right)^{1 - (\frac{1}{p_1} + \cdots + \frac{1}{p_m})} \leq 2^{(m-1)(1 - (\frac{1}{p_1} + \cdots + \frac{1}{p_m}))} \|T\|$$

for all m -linear forms $T : \ell_{p_1}^n \times \cdots \times \ell_{p_m}^n \rightarrow \mathbb{K}$ and all positive integers n . In particular, if $m < p \leq 2m$ then

$$\left(\sum_{j_1, \dots, j_m=1}^{\infty} |T(e_{j_1}, \dots, e_{j_m})|^{\frac{p}{p-m}} \right)^{\frac{p-m}{p}} \leq 2^{\frac{(m-1)(p-m)}{p}} \|T\|$$

for all continuous m -linear forms $T : \ell_p \times \cdots \times \ell_p \rightarrow \mathbb{K}$.

Proof. Let $S : \ell_{\infty}^n \times \cdots \times \ell_{\infty}^n \rightarrow \mathbb{K}$ be an m -linear form. Consider $s = \left(1 - \left(\frac{1}{p_1} + \cdots + \frac{1}{p_m}\right)\right)^{-1}$. Since $s \geq 2$, from Lemma 2, Hölder's inequality and Khinchin's inequality for multiple sums

([18]) we have

$$\begin{aligned}
& \sum_{j_1=1}^n \left(\sum_{\widehat{j}_1=1}^n |S(e_{j_1}, \dots, e_{j_m})|^s \right)^{\frac{1}{s}} \\
& \leq \sum_{j_1=1}^n \left(\left(\sum_{\widehat{j}_1=1}^n |S(e_{j_1}, \dots, e_{j_m})|^2 \right)^{\frac{1}{2}} \right)^{\frac{2}{s}} \left(\max_{\widehat{j}_1} |S(e_{j_1}, \dots, e_{j_m})| \right)^{1-\frac{2}{s}} \\
& \leq \sum_{j_1=1}^n \left(\left((\sqrt{2})^{m-1} R_n \right)^{\frac{2}{s}} R_n^{1-\frac{2}{s}} \right) \\
& = 2^{(m-1)\left(1-\left(\frac{1}{p_1}+\dots+\frac{1}{p_m}\right)\right)} \sum_{j_1=1}^n \int_{[0,1]^{m-1}} \left| \sum_{\widehat{j}_1=1}^n r_{j_2}(t_2) \cdots r_{j_m}(t_m) S(e_{j_1}, \dots, e_{j_m}) \right| dt_2 \cdots dt_m \\
& = 2^{(m-1)\left(1-\left(\frac{1}{p_1}+\dots+\frac{1}{p_m}\right)\right)} \int_{[0,1]^{m-1}} \sum_{j_1=1}^n \left| S \left(e_{j_1}, \sum_{j_2=1}^n r_{j_2}(t_2) e_{j_2}, \dots, \sum_{j_m=1}^n r_{j_m}(t_m) e_{j_m} \right) \right| dt_2 \cdots dt_m \\
& \leq 2^{\frac{(m-1)(p-m)}{p}} \sup_{t_2, \dots, t_m \in [0,1]} \sum_{j_1=1}^n \left| S \left(e_{j_1}, \sum_{j_2=1}^n r_{j_2}(t_2) e_{j_2}, \dots, \sum_{j_m=1}^n r_{j_m}(t_m) e_{j_m} \right) \right| \\
& \leq 2^{(m-1)\left(1-\left(\frac{1}{p_1}+\dots+\frac{1}{p_m}\right)\right)} \sup_{t_2, \dots, t_m \in [0,1]} \left\| S \left(\cdot, \sum_{j_2=1}^n r_{j_2}(t_2) e_{j_2}, \dots, \sum_{j_m=1}^n r_{j_m}(t_m) e_{j_m} \right) \right\| \\
& \leq 2^{(m-1)\left(1-\left(\frac{1}{p_1}+\dots+\frac{1}{p_m}\right)\right)} \|S\|,
\end{aligned}$$

where

$$R_n := \int_{[0,1]^{m-1}} \left| \sum_{\widehat{j}_1=1}^n r_{j_2}(t_2) \cdots r_{j_m}(t_m) S(e_{j_1}, \dots, e_{j_m}) \right| dt_2 \cdots dt_m.$$

Repeating the same procedure for other indexes we have

$$\sum_{j_i=1}^n \left(\sum_{\widehat{j}_i=1}^n |S(e_{j_1}, \dots, e_{j_m})|^s \right) \leq 2^{(m-1)\left(1-\left(\frac{1}{p_1}+\dots+\frac{1}{p_m}\right)\right)} \|S\|$$

for all $i = 1, \dots, m$. Hence, from Lemma 1, item (a), we conclude that

$$\left(\sum_{j_1, \dots, j_m=1}^n |T(e_{j_1}, \dots, e_{j_m})|^s \right)^{\frac{1}{s}} \leq 2^{(m-1)\left(1-\left(\frac{1}{p_1}+\dots+\frac{1}{p_m}\right)\right)} \|T\|$$

for all m -linear forms $T : \ell_{p_1}^n \times \cdots \times \ell_{p_m}^n \rightarrow \mathbb{K}$ and all positive integers n . \square

In particular the above result shows that for $m < p \leq m + c$ for a certain fixed constant c , we have a kind of uniform Hardy–Littlewood inequality, in the sense that there exists a universal constant, independent of m , satisfying the respective inequalities. For instance, if $c = 1$ we have

$$\left(\sum_{j_1, \dots, j_m=1}^{\infty} |T(e_{j_1}, \dots, e_{j_m})|^{\frac{p}{p-m}} \right)^{\frac{p-m}{p}} \leq 2^{\frac{m-1}{m+1}} \|T\| < 2 \|T\|$$

for all continuous m -linear forms $T : \ell_p \times \cdots \times \ell_p \rightarrow \mathbb{K}$ with $m < p \leq m + 1$.

If $p \leq 2m - 2$ we are able to improve exponents and constants:

Theorem 3. *Let $m \geq 2$ be a positive integer and $m < p \leq 2m - 2$. Then, for all continuous m -linear forms $T : \ell_p \times \cdots \times \ell_p \rightarrow \mathbb{K}$, we have*

$$\left(\sum_{j_i=1}^n \left(\sum_{\widehat{j_i=1}}^n |T(e_{j_1}, \dots, e_{j_m})|^{\frac{p}{p-(m-1)}} \right)^{\frac{p-(m-1)}{p} \cdot \frac{p}{p-m}} \right)^{\frac{p-m}{p}} \leq 2^{\frac{(m-1)(p-m+1)}{p}} \|T\|.$$

Proof. Consider $s = \frac{p}{p-(m-1)}$. Since $p \leq 2m - 2$ we have $s \geq 2$.

From Lemma 2, Hölder's inequality and Khinchin's inequality for multiple sums ([18]) we have, as in the proof of Theorem 2,

$$\sum_{j_i=1}^n \left(\sum_{\widehat{j_i=1}}^n |T(e_{j_1}, \dots, e_{j_m})|^{\frac{p}{p-(m-1)}} \right)^{\frac{p-(m-1)}{p}} \leq 2^{\frac{(m-1)(p-m+1)}{p}} \|T\|$$

for all m -linear forms $T \in \mathcal{L}(^m \ell_\infty^n; \mathbb{K})$ and all positive integers n . Note that

$$\left[\frac{1}{\lambda_0} - \sum_{j=1}^{m-1} \left(\frac{1}{p_j} - \frac{1}{q_j} \right) \right]^{-1} = \frac{1}{1 - \frac{m-1}{p}} = \frac{p}{p-(m-1)} = s.$$

From Lemma 1, item (b), we conclude that

$$\left(\sum_{j_i=1}^n \left(\sum_{\widehat{j_i=1}}^n |T(e_{j_1}, \dots, e_{j_m})|^{\frac{p}{p-(m-1)}} \right)^{\frac{p-(m-1)}{p} \cdot \frac{p}{p-m}} \right)^{\frac{p-m}{p}} \leq 2^{\frac{(m-1)(p-m+1)}{p}} \|T\|.$$

for all continuous m -linear forms $T : \ell_p \times \cdots \times \ell_p \rightarrow \mathbb{K}$. □

If we have the additional hypothesis that

$$\frac{1}{2} \leq \frac{1}{p_1} + \frac{1}{p_2} < 1$$

the optimal constants are always bounded by $2^{1 - (\frac{1}{p_1} + \frac{1}{p_2})}$:

Theorem 4. *Let $m \geq 3$ and $p_1, \dots, p_m \in (1, \infty]$ be such that*

$$\frac{1}{2} \leq \frac{1}{p_1} + \frac{1}{p_2} < 1$$

and

$$\frac{1}{p_1} + \cdots + \frac{1}{p_m} < 1.$$

Then

$$\left(\sum_{j_m=1}^n \left(\sum_{\widehat{j_m=1}}^n |T(e_{j_1}, \dots, e_{j_m})|^{\frac{1}{1 - (\frac{1}{p_1} + \cdots + \frac{1}{p_{m-1}})}} \right)^{\frac{1 - (\frac{1}{p_1} + \cdots + \frac{1}{p_{m-1}})}{1 - (\frac{1}{p_1} + \cdots + \frac{1}{p_m})}} \right)^{1 - (\frac{1}{p_1} + \cdots + \frac{1}{p_m})} \leq 2^{1 - (\frac{1}{p_1} + \frac{1}{p_2})} \|T\|$$

for all m -linear forms $T : \ell_{p_1}^n \times \cdots \times \ell_{p_m}^n \rightarrow \mathbb{K}$ and all positive integers n .

Proof. Since $\frac{1}{p_1} + \frac{1}{p_2} \geq \frac{1}{2}$ by Theorem 2 we have

$$\left(\sum_{i,j=1}^n |T_2(e_i, e_j)|^{\frac{1}{1-\left(\frac{1}{p_1}+\frac{1}{p_2}\right)}} \right)^{1-\left(\frac{1}{p_1}+\frac{1}{p_2}\right)} \leq 2^{1-\left(\frac{1}{p_1}+\frac{1}{p_2}\right)} \|T\|$$

for all bilinear forms $T_2 : \ell_{p_1}^n \times \ell_{p_2}^n \rightarrow \mathbb{K}$ and all positive integers n . By the Khinchin inequality we conclude that

$$\left(\sum_{i,j=1}^n \left(\sum_{k=1}^n |T_3(e_i, e_j, e_k)|^2 \right)^{\frac{1}{2} \cdot \frac{1}{1-\left(\frac{1}{p_1}+\frac{1}{p_2}\right)}} \right)^{1-\left(\frac{1}{p_1}+\frac{1}{p_2}\right)} \leq 2^{1-\left(\frac{1}{p_1}+\frac{1}{p_2}\right)} \|T\|$$

for all 3-linear forms $T_3 : \ell_{p_1}^n \times \ell_{p_2}^n \times \ell_{\infty}^n \rightarrow \mathbb{K}$ and all positive integers n . In fact, for all positive integers n we have

$$\begin{aligned} & \left(\sum_{i,j=1}^n \left(\sum_{k=1}^n |T_3(e_i, e_j, e_k)|^2 \right)^{\frac{1}{2} \cdot \frac{1}{1-\left(\frac{1}{p_1}+\frac{1}{p_2}\right)}} \right)^{1-\left(\frac{1}{p_1}+\frac{1}{p_2}\right)} \\ & \leq A^{-\frac{1}{1-\left(\frac{1}{p_1}+\frac{1}{p_2}\right)}} \left(\sum_{i,j=1}^n \int_0^1 \left| \sum_{k=1}^n r_k(t) T_3(e_i, e_j, e_k) \right|^{\frac{1}{1-\left(\frac{1}{p_1}+\frac{1}{p_2}\right)}} dt \right)^{1-\left(\frac{1}{p_1}+\frac{1}{p_2}\right)} \\ & = \left(\int_0^1 \sum_{i,j=1}^n \left| T_3 \left(e_i, e_j, \sum_{k=1}^n r_k(t) e_k \right) \right|^{\frac{1}{1-\left(\frac{1}{p_1}+\frac{1}{p_2}\right)}} dt \right)^{1-\left(\frac{1}{p_1}+\frac{1}{p_2}\right)} \\ & \leq \sup_{t \in [0,1]} \left(\sum_{i,j=1}^n \left| T_3 \left(e_i, e_j, \sum_{k=1}^n r_k(t) e_k \right) \right|^{\frac{1}{1-\left(\frac{1}{p_1}+\frac{1}{p_2}\right)}} \right)^{1-\left(\frac{1}{p_1}+\frac{1}{p_2}\right)} \\ & \leq 2^{1-\left(\frac{1}{p_1}+\frac{1}{p_2}\right)} \|T_3\| \end{aligned}$$

Thus, since $\left(1 - \left(\frac{1}{p_1} + \frac{1}{p_2}\right)\right)^{-1} \geq 2$,

$$\left(\sum_{i,j,k=1}^n |T_3(e_i, e_j, e_k)|^{\frac{1}{1-\left(\frac{1}{p_1}+\frac{1}{p_2}\right)}} \right)^{1-\left(\frac{1}{p_1}+\frac{1}{p_2}\right)} \leq 2^{1-\left(\frac{1}{p_1}+\frac{1}{p_2}\right)} \|T_3\|$$

for all 3-linear forms $T_3 : \ell_{p_1}^n \times \ell_{p_2}^n \times \ell_{\infty}^n \rightarrow \mathbb{K}$ and all positive integers n . This means that for any Banach spaces E_1, E_2, E_3 , every continuous 3-linear form $R : E_1 \times E_2 \times E_3 \rightarrow \mathbb{K}$ is multiple $\left(\frac{1}{1-\left(\frac{1}{p_1}+\frac{1}{p_2}\right)}; p_1^*, p_2^*, 1\right)$ -summing (see [10]). By the essence of the inclusion theorem for multiple summing operators proved in [17], since

$$\frac{1}{1} - \frac{1}{\frac{1}{1-\left(\frac{1}{p_1}+\frac{1}{p_2}\right)}} = \frac{1}{p_3^*} - \frac{1}{\frac{1}{1-\left(\frac{1}{p_1}+\frac{1}{p_2}+\frac{1}{p_3}\right)}},$$

with $E_1 = \ell_{p_1}^n$, $E_2 = \ell_{p_2}^n$ and $E_3 = \ell_{p_3}^n$, we conclude that

$$\begin{aligned} & \left(\sum_{k=1}^n \left(\sum_{i,j=1}^n |S(e_i, e_j, e_k)|^{\frac{1}{1-(\frac{1}{p_1} + \frac{1}{p_2})}} \right)^{1-(\frac{1}{p_1} + \frac{1}{p_2})} \cdot \frac{1}{1-(\frac{1}{p_1} + \frac{1}{p_2} + \frac{1}{p_3})} \right)^{1-(\frac{1}{p_1} + \frac{1}{p_2} + \frac{1}{p_3})} \\ & \leq 2^{1-(\frac{1}{p_1} + \frac{1}{p_2})} \|S\| \end{aligned}$$

for all 3-linear forms $S : \ell_{p_1}^n \times \ell_{p_2}^n \times \ell_{p_3}^n \rightarrow \mathbb{K}$ and all positive integers n . The proof is completed by a standard induction argument. \square

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